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Supersymmetric quantum mechanics of one-dimensional systems

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Abstract. It is shown that every one-dimensional quantum mechanical Hamiltonian H_1 can have a partner H_2 such that H_1 and H_2 taken together may be viewed as the components of a supersymmetric Hamiltonian. The term 'supersymmetric Hamiltonian' is taken to mean a Hamiltonian defined in terms of charges that obey the same algebra as that of the generators of supersymmetry in field theory. The consequences of this symmetry for the spectra of H_1 and H_2 are explored. It is shown how the supersymmetric pairing may be utilised to eliminate the ground state of H_1, or add a state below the ground state of H_1 or maintain the spectrum of H_1. It is also explicitly demonstrated that the supersymmetric pairing may be used to generate a class of anharmonic potentials with exactly specified spectra. The complete spectrum of an anharmonic potential so generated consists of all the eigenstates of the simple harmonic oscillator and, in addition, a ground state at a specified energy E which lies arbitrarily below the $E=\frac{1}{2}$ ground state of the harmonic oscillator.

1. Introduction

In field theory, supersymmetry is a symmetry that generates transformations between bosons and fermions. The generators of the supersymmetric transformations are spinor charges whose algebra involves anticommutators, unlike the generators of other symmetries whose algebra involves commutators. Supersymmetry raises the possibility of providing a framework for a unified description of bosons and fermions which are combined in the same supersymmetric multiplet (Wess and Zumino 1974). Furthermore, supersymmetric field theories are found to be much less divergent than ordinary field theories. These attractive features have aroused enormous interest in supersymmetric field theories. A full understanding of such theories requires familiarity with Grassmann algebra, the algebra of anticommuting quantities. It has been shown that supersymmetric field theories may be constructed by defining a superfield in a superspace, a space consisting of the usual spacetime and in addition the anticommuting spinors of Grassmann (Salam and Strathdee 1975). The superfield φ is then a function of spacetime coordinates x and also of θ and $\bar{\theta}$ where θ is an odd member of the Grassmann algebra and $\bar{\theta}$ is its conjugate. The supersymmetric transformation may be viewed as a Grassmann even translation in this superspace. The generators of this transformation are the supercharges. Supersymmetric field theories may also be defined without explicit reference to superspace or to Grassmann numbers.

In an important paper on supersymmetry, Witten (1981) constructed as a simple example of a supersymmetric system, a spin-$\frac{1}{2}$ particle moving in one dimension. This

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was an example of a supersymmetric system with no reference to a field theory at all! Witten also defined the algebra that must be satisfied by the charge operators in terms of which the supersymmetric Hamiltonian may be expressed. These algebraic relations that Witten first formulated have now become the defining relations of supersymmetric quantum mechanics. The connection of these relations to the field theoretical counterparts may also be established. The arguments given below are those of Freedman and Cooper (1983) and Takeda and Ui (1984). Quantum mechanics in one dimension is defined by the Hamiltonian $\frac{1}{2}p^2 + V(x)$ and the commutation relation $[x, p] = i\hbar$. Field theory starts by defining a spacetime and the field $\varphi(x, t)$ is defined in this spacetime. The dynamics of the system is determined by an action. In $d = 1$, the action S is given by $S = \int dt[(\partial_t\varphi)^2 - V(\varphi)]$. It is well known that $d = 1$ quantum mechanics is formally equivalent to the $d = 1$ quantum field theory with the identification $x \leftrightarrow \varphi$, $p \leftrightarrow \partial_t\varphi$ and canonical quantisation of the field φ leads to the usual commutation relation between x and p. Similarly, by constructing a Lagrangian invariant under the supersymmetric transformation, i.e. by generalising the $d = 1$ field to the superfield defined in superspace and integrating out the Grassmann coordinates associated with the superspace a Lagrangian expressed in terms of the component fields of the superfield is obtained. Canonical quantisation then leads to the supersymmetric quantum mechanical Hamiltonian first defined by Witten (1981).

The word 'supersymmetry' was originally used to denote a symmetry built into certain field theories that permits transformations between the component fields whose intrinsic spins differ by $\frac{1}{2}\hbar$. In this paper, the term 'supersymmetry' is used in a general sense; none of the systems referred to in this paper have anything to do with spins at all. The term 'supersymmetric system' is used to describe systems governed by an underlying algebra which is identical to or derivable from the algebra of supersymmetry in field theory. This algebra is the algebra explicitly defined by Witten.

Just as in field theory, supersymmetry leads to specific relations between the component sectors of a supermultiplet, so also in supersymmetric quantum mechanics, the existence of a symmetry generating operator that commutes with the Hamiltonian leads to certain specific relations between the spectra of the component parts of the supersymmetric Hamiltonian. This is the major reason for the upsurge of interest in supersymmetric quantum mechanics (Blockley and Stedman 1985, Ui 1984, Kosteleckey and Nieto 1984, Yamagishi 1984). Bernstein and Brown (1984) showed that by exploiting the degeneracy between the spectra of the 'bosonic' and 'fermionic' sector of *certain* one-dimensional supersymmetric Hamiltonians, the properties of the first excited state of the bosonic component may be inferred from the knowledge of the ground state of the fermionic component. It has been shown recently by Andrianov *et al* (1984) and Sukumar (1985a) independently that *all* one-dimensional quantum systems have supersymmetric partners. Andrianov *et al* have also shown that a simple extension of supersymmetric quantum mechanics to arbitrary dimensions is possible.

In an earlier communication (Sukumar 1985a) it was shown that the existence of a supersymmetric partner to every one-dimensional Schrödinger equation implies the existence of a hierarchy of Hamiltonians with a special relationship between the eigenvalues and eigenfunctions of the different members of the hierarchy. In this paper, we present a full version of the properties of supersymmetric partners in one dimension. In the accompanying paper (Sukumar 1985b, hereafter referred to as II) we study how some aspects of the inverse scattering theory can be understood using the concept of a supersymmetric partner to a Hamiltonian. The plan of this paper is as follows: § 2 presents the defining algebra of supersymmetric quantum mechanics

and the implications of this algebra for the spectra of the component parts of the supersymmetric Hamiltonian. The factorisation of the Schrödinger equation is discussed in § 3. The procedure for the elimination of the ground state of a Hamiltonian is discussed in § 4. Section 5 shows how supersymmetry may be used to introduce new bound states to a given Hamiltonian or generate a new Hamiltonian with unaltered spectrum. Section 6 presents examples to illustrate the procedure outlined in § 5 and § 7 contains the conclusions.

2. Supersymmetric quantum mechanics

According to Witten, supersymmetric quantum mechanics is characterised by the existence of charge operators Q_i that obey the algebra

$$\{Q_i, Q_J\} = \delta_{iJ}\mathscr{H} \qquad i, J = 1, 2, \ldots N \tag{1}$$

$$[Q_i, \mathscr{H}] = 0 \tag{2}$$

where $\mathscr{H}$ is the supersymmetric Hamiltonian, N is the number of generators and $\{\ldots, \ldots\}$ denotes an anticommutator. In this paper, we consider the simplest of such systems with two operators Q_1 and Q_2. In terms of $Q = (Q_1 + iQ_2)/\sqrt{2}$ and its adjoint $Q^\dagger = (Q_1 - iQ_2)/\sqrt{2}$ the algebra governing this supersymmetric system can be characterised by

$$\mathscr{H} = \{Q, Q^\dagger\} \qquad Q^2 = 0 \quad \text{and} \quad Q^{\dagger 2} = 0. \tag{3}$$

From these equations, it is easy to see that

$$[Q, \mathscr{H}] = 0 \qquad \text{and} \qquad [Q^\dagger, \mathscr{H}] = 0 \tag{4}$$

i.e. the charge operator Q is nil-potent and commutes with the Hamiltonian $\mathscr{H}$. A simple realisation of the algebra defined in equation (3) can be achieved by considering

$$Q = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix} \qquad \text{and} \qquad Q^\dagger = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} \tag{5}$$

where A^- is a linear differential operator, A^+ is the adjoint and $Q^2 = 0$ by construction. This construction does not make any assumptions about the commutator $[A^+, A^-]$. In particular, the commutator of A^+ and A^- is not restricted to be a constant as in the case of the simple harmonic oscillator. Equation (5) leads to the supersymmetric Hamiltonian

$$\mathscr{H} = \begin{pmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{pmatrix}. \tag{6}$$

Since

$$Q\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ A^-\alpha \end{bmatrix} \qquad \text{and} \qquad Q^\dagger\begin{bmatrix} 0 \\ \beta \end{bmatrix} = \begin{bmatrix} A^+\beta \\ 0 \end{bmatrix} \tag{7}$$

we can say that the operators Q and $Q^\dagger$ induce transformations between the 'bosonic' sector represented by α and the 'fermionic' sector represented by β. We may also interpret $\mathscr{H}$ in the following way: the scalar Hamiltonian $H_1 = A^+A^-$ has a partner $H_2 = A^-A^+$ such that H_1 and H_2 are the diagonal elements of a supersymmetric Hamiltonian $\mathscr{H}$. Having demonstrated that a Q and an $\mathscr{H}$ can be constructed, we switch to the operator language of quantum mechanics to find out what the consequences of the existence of a charge operator that commutes with the Hamiltonian are for the spectra of the two sectors H_1 and H_2.

A^+A^- and A^-A^+ are both positive semi-definite operators with eigenvalues greater than or equal to 0. Let ψ_1 be a normalised eigenstate of H_1 with eigenvalue E_1. Then

$$A^+A^-\psi_1 = E_1\psi_1. \tag{8}$$

Multiplication from the left by A^- leads to

$$A^-A^+(A^-\psi_1) = E_1(A^-\psi_1). \tag{9}$$

If $A^-\psi_1 \neq 0$, we can infer that E_1 is also an eigenvalue of A^-A^+. The corresponding normalised eigenfunction ψ_2 of H_2 is given by

$$\psi_2 = (E_1)^{-1/2}(A^-\psi_1). \tag{10}$$

The same reasoning may be applied starting from the eigenvalue equation for H_2 instead of H_1 to investigate whether every eigenvalue of H_2 is also an eigenvalue of H_1. If E_2 is an eigenvalue of H_2 with eigenfunction ψ_2,

$$A^-A^+\psi_2 = E_2\psi_2 \tag{11}$$

then

$$A^+A^-(A^+\psi_2) = E_2(A^+\psi_2). \tag{12}$$

Therefore if $A^+\psi_2 \neq 0$, then E_2 is also an eigenvalue of H_1 with the corresponding normalised eigenfunction

$$\psi_1 = (E_2)^{-1/2}(A^+\psi_2). \tag{13}$$

In view of the above relationships, three possibilities can be distinguished from each other.

(*a*) If there is a normalisable eigenstate of H_1 such that $A^-\psi_1^{(0)} = 0$, then $A^+A^-\psi_1^{(0)} = 0$ and $\psi_1^{(0)}$ corresponds to the ground state with eigenvalue $E_1^{(0)} = 0$. Conversely, for the eigenvalue $E_1^{(0)} = 0$, the vanishing expectation value $\langle\psi_1^{(0)}|A^+A^-|\psi_1^{(0)}\rangle$ implies that $A^-\psi_1^{(0)} = 0$. Under these circumstances, H_2 has no normalisable eigenstate with $E_2 = 0$, i.e. there can be no normalisable state with $A^+\psi_2 = 0$. The ground-state eigenvalue of H_2 is non-zero. All eigenvalues other than the ground-state eigenvalue of H_1 are also eigenvalues of H_2 and all eigenvalues of H_2 are also eigenvalues of H_1. The resulting spectral mapping is shown in figure 1(*a*).

(*b*) If there is a normalisable eigenstate of H_2 such that $A^+\psi_2^{(0)} = 0$, then $A^-A^+\psi_2^{(0)} = 0$ and $\psi_2^{(0)}$ corresponds to the ground state of A^-A^+ with eigenvalue $E_2^{(0)} = 0$. There cannot be a normalisable state of A^+A^- with eigenvalue zero that satisfies $A^-\psi_1 = 0$. The ground state of H_1 has non-zero eigenvalue. All eigenvalues other than the ground state of H_2 are also eigenvalues of H_1, and all eigenvalues of H_1 are eigenvalues of H_2. This leads to the spectral mapping shown in figure 1(*b*).

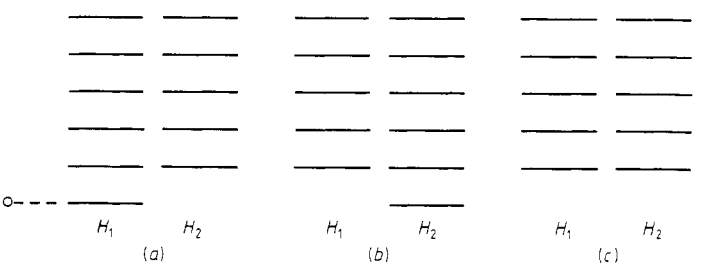

Figure 1. Schematic diagram of the possible alignment of the eigenvalues of the operators $H_1 = A^+A^-$ and $H_2 = A^-A^+$.

(*c*) If there is no normalisable eigenstates of H_1 or H_2 such that either $A^-\psi_1 = 0$ or $A^+\psi_2 = 0$, then the spectra of both H_1 and H_2 begin at positive values. Every eigenvalue of H_1 is then also an eigenvalue of H_2 and vice versa. The resulting spectral mapping for this case is shown in figure 1(*c*).

In each of the cases (*a*), (*b*) and (*c*), the eigenfunctions of H_1 and H_2 for a common eigenvalue E are linked in the manner indicated below:

$$\psi_2(E) = \exp(i\varphi)(E)^{-1/2}A^-\psi_1(E) \tag{14a}$$

$$\psi_1(E) = \exp(-i\varphi)(E)^{-1/2}A^+\psi_1(E) \tag{14b}$$

in which φ is an arbitrary phase whose significance will become clear in later discussion. The ladder structure of the eigenvalue spectrum shown in figure 1 and the intertwining relationship between the eigenfunctions in equation (14) are characteristic hallmarks of supersymmetric systems in one dimension and serve as the signatures by which an underlying supersymmetry may be inferred.

Bernstein and Brown (1984) and Nieto (1984) considered operators of the form

$$A^{\pm} = [\pm \mathrm{d}/\mathrm{d}x + \tilde{V}(x)] \tag{15}$$

where $V(x)$ was assumed to be a known function of x. This assumption restricts the applicability of supersymmetric quantum mechanics to a limited class of problems. In the next section, we show that it is not necessary to assume that $V(x)$ is known and that V may itself be generated from the solutions of the Schrödinger equation in one dimension. Such a generalisation then extends the applicability of supersymmetric quantum mechanics to all one-dimensional problems (Andrianov *et al* 1984, Sukumar 1985a).

3. Factorisation of the Schrödinger equation

The Schrödinger equation in n dimensions is governed by the Hamiltonian

$$H = -\tfrac{1}{2}\nabla^2 + V(x_1, \ldots x_n) \tag{16}$$

where ∇^2 is the Laplacian in n dimensions and V is the potential. H can always be factorised in the form

$$H = \sum_n A_n^+ A_n^- + \mathscr{E} \tag{17a}$$

with

$$A_n^{\pm} = (1/\sqrt{2})(\pm\nabla_n + U_n) \tag{17b}$$

∇_n the gradient operator and $\mathscr{E}$ an undetermined constant, provided that the unknown function U_n satisfies

$$\sum_n (\nabla_n U_n + U_n^2) = 2(V - \mathscr{E}). \tag{18}$$

This is a nonlinear equation with a family of solutions. One member of the family is given by

$$U_n = \frac{1}{\psi(\mathscr{E})}\nabla_n \psi(x, \mathscr{E}) \tag{19}$$

where $\psi(\mathscr{E})$ is a solution of the Schrödinger equation at energy $E = \mathscr{E}$, i.e.

$$H\psi = \mathscr{E}\psi. \tag{20}$$

It can easily be verified that (19) satisfies (18). Since

$$\nabla_n U_n = \frac{1}{\psi}\nabla_n^2\psi - \frac{1}{\psi^2}(\nabla_n\psi)^2$$
$$\sum_n (\nabla_n U_n + U_n^2) = \frac{\nabla^2\psi}{\psi} = 2(V-\mathscr{E}). \tag{21}$$

It is clear that this argument is valid only if $\psi(x, \mathscr{E})$ is non-vanishing, i.e. $\psi(x, \mathscr{E})$ is nodeless. In the rest of this paper we consider only the solutions in one dimension.

It is shown in the appendix that in one dimension the family of solutions to the nonlinear equation (18) is given by

$$U(x, \mathscr{E}, \lambda) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\psi(x, \mathscr{E}) + \frac{1/\psi^2(x, \mathscr{E})}{(\lambda + \int^x \mathrm{d}z/\psi^2(z, \mathscr{E}))} \tag{22}$$

where λ is an arbitrary parameter. Every choice of energy $\mathscr{E}$ and the corresponding $\psi(x, \mathscr{E})$ leads to a possible factorisation of H in the form $H = A^+(\mathscr{E})A^-(\mathscr{E}) + \mathscr{E}$. The choice of the factorisation energy $\mathscr{E}$ and the selection of a member from the family of solutions $U(\mathscr{E}, \lambda)$ must clearly be motivated by the particular circumstances of a given problem and by physical considerations.

A distinction between the supersymmetric field theory and the supersymmetry of the Hamiltonians considered in this paper must be made. In supersymmetric field theory, the supersymmetric Hamiltonian is required to be positive semi-definite and the ground state must have zero energy. In supersymmetric quantum mechanics, the operators A^+A^- and A^-A^+ are required to be positive definite, but the Hamiltonians $H_1 = A^+(\mathscr{E})A^-(\mathscr{E}) + \mathscr{E}$ and $H_2 = A^-(\mathscr{E})A^+(\mathscr{E}) + \mathscr{E}$ are not required to be positive definite because $\mathscr{E}$ occurs as an arbitrary non-positive definite factorisation energy. This is related to the fact that in quantum mechanics, in contrast to quantum field theory, the origin of the energy scale is arbitrary. Different choices of $\mathscr{E}$ lead to different operators $A(\mathscr{E})$ and the relation between the spectra of H_1 and H_2 crucially depends upon the factorisation energy $\mathscr{E}$, as the following analysis will show.

If we consider the factorisation $H = A^+(\mathscr{E})A^-(\mathscr{E}) + \mathscr{E}$ for a Hamiltonian with ground state at energy $E^{(0)}$, then the requirement that A^+A^- be a positive semi-definite operator can be met only if the energy $\mathscr{E}$ is chosen to be $\mathscr{E} \leq E^{(0)}$. We consider the case when the factorisation energy $\mathscr{E} = E^{(0)}$ in the next section.

4. Factorisation energy $\mathscr{E}$ equals ground-state energy $E^{(0)}$

With the choice of $E^{(0)}$ as the factorisation energy, the ground-state wavefunction $\psi(x, E^{(0)})$ is nodeless and vanishes in the asymptotic region $|x| \to \infty$. The requirement that $U(x)$ in equation (22) should not be divergent leads to the choice $\lambda = \infty$, giving

$$U(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\psi(x, E^{(0)}) \tag{23a}$$

$$A^{\pm}(E^{(0)}) = \frac{1}{\sqrt{2}}\left[\pm\frac{\mathrm{d}}{\mathrm{d}x} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\ln\psi(x, E^{(0)})\right)\right] \tag{23b}$$

$$H = A^+(E^{(0)})A^-(E^{(0)}) + E^{(0)}. \tag{23c}$$

It is clear that A^+A^- has a spectrum beginning at 0, with a ground state which satisfies $A^-\psi^{(0)}=0$ with $\psi^{(0)}=\psi(x, E^{(0)})$. The analysis of § 2 can now be used by considering the partner Hamiltonian

$$\tilde{H}=A^-(E^{(0)})A^+(E^{(0)})+E^{(0)}=H+[A^-(E^{(0)}), A^+(E^{(0)})] \tag{24a}$$

corresponding to the potential

$$\tilde{V}(x)=V(x)-\frac{d^2}{dx^2}\ln\psi^{(0)}. \tag{24b}$$

H and $\tilde{H}$ must have their spectra aligned as in figure 1(a). $\tilde{H}$ has no eigenstate corresponding to the ground state of H and all the excited states of H are degenerate with the eigenstates of $\tilde{H}$. The eigenfunctions of the two Hamiltonians are linked in the form

$$\tilde{\psi}(x, E)=(E-E^{(0)})^{-1/2}A^-(E^{(0)})\psi(x, E) \tag{25a}$$

$$\psi(x, E)=(E-E^{(0)})^{-1/2}A^+(E^{(0)})\tilde{\psi}(x, E) \tag{25b}$$

by choosing the phase φ in equation (14) to be zero. These equations are valid not only when E is one of the discrete eigenstates of H, $E=E^{(J)}(J\neq 0)$, but also when E lies in the continuous part of the spectrum. When E lies in the continuous part of the spectrum of H, the above equations can be used to find a relation between the transmission coefficients in the potentials $V(x)$ and $\tilde{V}(x)$ at energy E, since the asymptotic form of the wavefunction for potential V at energy E implies a definite asymptotic form for the wavefunction for $\tilde{V}$ at the same energy. The procedure will be illustrated by considering the phaseshifts for the solutions of the radial Schrödinger equation in II.

Since the above analysis is valid for any one-dimensional Hamiltonian H_1 with ground state $[E_1^{(0)}, \psi_1^{(0)}]$ the process of finding a supersymmetric partner can be iterated (Sukumar 1985a) to generate the hierarchy of Hamiltonians

$$H_n(x)=-\frac{1}{2}\frac{d^2}{dx^2}+V_n(x)\equiv A_n^+A_n^-+E_n^{(0)}=A_{n-1}^-A_{n-1}^++E_{n-1}^{(0)} \qquad n=2, 3\ldots \tag{26a}$$

where

$$A_n^{\pm}(x)=\frac{1}{\sqrt{2}}\left[\pm\frac{d}{dx}+\left(\frac{d}{dx}\ln\psi_n^{(0)}(x)\right)\right] \qquad n=1, 2, \ldots \tag{26b}$$

$$V_n(x)=V_{n-1}(x)-\frac{d^2}{dx^2}\ln\psi_{n-1}^{(0)}(x) \qquad n=2, 3, \ldots \tag{26c}$$

in which $[E_n^{(J)}, \psi_n^{(J)}]$ are the eigenstates of H_n with the property that

$$E_n^{(m)}=E_{n-1}^{(m+1)}=\ldots=E_1^{(m+n-1)} \tag{27a}$$

$$\psi_n^{(m)}=[E_n^{(m)}-E_{n-1}^{(0)}]^{-1/2}A_{n-1}^-\psi_{n-1}^{(m+1)} \tag{27b}$$

$$\psi_{n-1}^{(m+1)}=[E_n^{(m)}-E_{n-1}^{(0)}]^{-1/2}A_{n-1}^+\psi_n^{(m)} \qquad n=2, 3, \ldots; \qquad m=0, 1, 2, \ldots. \tag{27c}$$

A pictorial representation of the eigenvalue correspondence of the Hamiltonian hierarchy is given in figure 2. The equations given above show that the excited states of V_1 can be obtained from the ground states of the hierarchy V_n. Several examples of this Hamiltonian hierarchy have already been given (Sukumar 1985a). Here we discuss one more example of the hierarchy H_n.

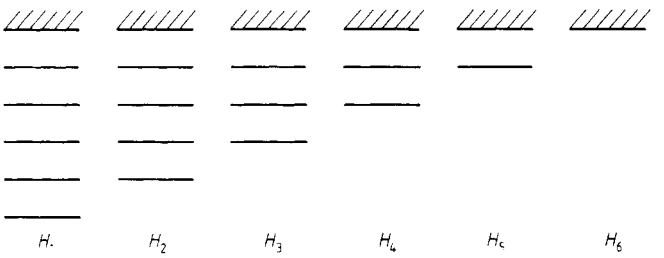

Figure 2. Schematic diagram of the eigenvalue spectra of the Hamiltonians in the hierarchy H_n. The number of bound states of H_1 is arbitrarily chosen to be 5.

4.1. Attractive sech^2 x potential

Let

$$V_1 = -\lambda_1 \operatorname{sech}^2 x \qquad \lambda_1 > 0. \tag{28}$$

Since this potential is attractive in all space $-\infty \leqslant x \leqslant \infty$ it will support at least one bound state irrespective of the strength of the potential. In terms of the parameter

$$Q_1 = (2\lambda_1 + \tfrac{1}{4})^{1/2} \geqslant \tfrac{1}{2} \tag{29}$$

the spectrum of this potential is given by (Morse and Feshbach 1953)

$$E_1^{(m)} = -\tfrac{1}{2}[Q_1 - (m + \tfrac{1}{2})]^2 \qquad m = 0, 1, 2, \ldots N \leqslant (Q_1 - \tfrac{1}{2}). \tag{30}$$

The potential (28) supports a finite number $(N+1)$ of bound states. The ground-state wavefunction

$$\psi_1^{(0)}(x) = (\operatorname{sech} x)^{(Q_1 - \frac{1}{2})} \tag{31}$$

leads to

$$V_2(x) = -(\lambda_1 + \tfrac{1}{2} - Q_1) \operatorname{sech}^2 x. \tag{32}$$

Inspection of this equation shows that

(i) if $\lambda_1 > 1$, then $V_2(x)$ is an attractive sech$^2 x$ potential;
(ii) if $\lambda_1 = 1$, $V_2(x)$ vanishes and H_2 is a free particle Hamiltonian;
(iii) if $\lambda_1 < 1$, $V_2(x)$ is repulsive and corresponds to a sech^2 barrier.

It is easy to show that the parameter corresponding to Q_1 for V_2 is

$$Q_2 = [\tfrac{1}{4} + 2(\lambda_1 + \tfrac{1}{2} - Q_1)]^{1/2} = Q_1 - 1. \tag{33}$$

The spectrum of H_2 is then given by

$$E_2^{(m)} = -\tfrac{1}{2}[Q_2 - (m + \tfrac{1}{2})]^2 \tag{34}$$

which satisfies the condition $E_2^{(m)} = E_1^{(m+1)}$. Iteration of this argument shows that the Hamiltonian hierarchy corresponds to a sequence of sech^2 potentials with successively decreasing strengths. It is easy to show that

$$\begin{aligned} &V_n = -\lambda_n \operatorname{sech}^2 x \qquad Q_n = (2\lambda_n + \tfrac{1}{4})^{1/2} \\ &Q_n = Q_{n-1} - 1. \end{aligned} \tag{35}$$

If $N = Q_1 - \frac{1}{2}$, $V_{n+1}(x)$ vanishes. If $N < (Q_1 - \frac{1}{2})$, $V_{N+2}(x)$ corresponds to a sech^2 barrier, given by

$$V_{N+2}(x) = \tfrac{1}{2}(Q_1 - N - \tfrac{1}{2})(N + \tfrac{3}{2} - Q_1) \operatorname{sech}^2 x. \tag{36}$$

In this section, we have shown that by choosing the factorisation energy $\mathscr{E}$ to be the ground-state energy it is possible to generate a new Hamiltonian H_2 without an eigenstate at the energy corresponding to the ground state of H_1, but retaining the rest of the spectrum of H_1. It has been demonstrated that this procedure may be iterated to generate a Hamiltonian hierarchy with spectra aligned as in figure 2. Other possible factorisations are examined in the next section.

5. Factorisation energy $\mathscr{E}$ less than the ground-state energy

When the factorisation energy $\mathscr{E}$ in equation (17) is less than the ground-state energy $E^{(0)}$ of H, the solution $\psi(x, \mathscr{E})$ of $H\psi = \mathscr{E}\psi$ is not a normalisable solution, even though $\psi(\mathscr{E})$ is still a solution of $A^-(\mathscr{E})\psi(\mathscr{E}) = 0$. The lack of normalisability of $\psi(\mathscr{E})$ means that $A^+(\mathscr{E})A^-(\mathscr{E})$ cannot have zero as an eigenvalue and the spectrum of A^+A^- begins at positive values. The analysis of § 2 shows that when A^+A^- has no normalisable eigenstate with eigenvalue zero, it is possible for A^-A^+ to have spectrum beginning at eigenvalue zero. For A^+A^- to have a normalisable state with eigenvalue zero, the solution $\tilde{\psi}$ of $A^+(\mathscr{E})\tilde{\psi} = 0$ must be normalisable. The solution of

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\ln\psi(x, \mathscr{E})\right)\right]\tilde{\psi}(x, \mathscr{E}) = 0 \tag{37}$$

i.e.

$$\tilde{\psi}(x, \mathscr{E}) = 1/\psi(x, \mathscr{E}) \tag{38}$$

shows that if the unnormalisable solution $\psi(x, \mathscr{E})$ of the Hamiltonian H is chosen in such a way that $1/\psi$ is normalisable, then $\tilde{\psi}(x, \mathscr{E})$ is normalisable and A^-A^+ has a spectrum beginning at eigenvalue zero. Therefore

$$\tilde{H} = A^-(\mathscr{E})A^+(\mathscr{E}) + \mathscr{E} \qquad \mathscr{E} < E^{(0)} \tag{39}$$

has a ground state at energy $\tilde{E}^{(0)} = \mathscr{E}$ with a ground-state eigenfunction $\tilde{\psi}^{(0)}(x, \mathscr{E}) = \tilde{\psi}(x, \mathscr{E})$. $\tilde{H}$, therefore, has a ground-state eigenvalue below the ground state of H while all other eigenstates of $\tilde{H}$ are degenerate with the eigenstates of H. This corresponds to the level scheme depicted in figure 1(b). Hence when $1/\psi$ is normalisable

$$\tilde{H} = H - \frac{\mathrm{d}^2}{\mathrm{d}x^2}[\ln\psi(x, \mathscr{E})] \tag{40}$$

has ground state

$$\tilde{E}^{(0)} = \mathscr{E} < E^{(0)} \tag{41a}$$

$$\tilde{\psi}^{(0)}(x, \mathscr{E}) = 1/\psi(x, \mathscr{E}) \tag{41b}$$

and excited states with

$$\tilde{E}^{(m+1)} = E^{(m)} \qquad m = 0, 1, 2, \ldots, \tag{42a}$$

$$\tilde{\psi}^{(m+1)} = -(E^{(m)} - \mathscr{E})^{-1/2}A^-\psi^{(m)} \tag{42b}$$

$$\psi^{(m)} = -(E^{(m)} - \mathscr{E})^{-1/2}A^+\tilde{\psi}^{(m+1)} \tag{42c}$$

where

$$A^{\pm}(\mathscr{E}) = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}x} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\ln\psi(x, \mathscr{E})\right)\right]. \tag{42d}$$

The phase factor φ in equation (14) has been chosen to be π. Having chosen φ to be zero for the case of elimination of a state in § 3, the requirement that adding a state by a transformation and subsequently eliminating the same state by another transformation should give back the original eigenfunctions, fixes the phase factor for the case of addition of a state to be π.

If $\mathscr{E} < E^{(0)}$, but the unnormalisable solution $\psi(x, \mathscr{E})$ does not lead to a normalisable $1/\psi$ and $(d^2/dx^2) \ln \psi(x, \mathscr{E})$ is well behaved, in a sense to be defined shortly, then neither $A^+(\mathscr{E})A^-(\mathscr{E})$ nor $A^-(\mathscr{E})A^+(\mathscr{E})$ has a normalisable eigenstate with eigenvalue zero. We denote such a solution ψ by ξ. Therefore, A^+A^- and A^-A^+ have identical spectra as depicted in figure 1(*c*). Then

$$\tilde{H} = A^-(\mathscr{E})A^+(\mathscr{E}) + \mathscr{E} = H - \frac{d^2}{dx^2} \ln \xi(x, \mathscr{E}) \tag{43}$$

has a spectrum identical to that of H:

$$\tilde{E}^{(m)} = E^{(m)} \qquad m = 0, 1, 2, \ldots . \tag{44a}$$

The eigenfunction relations are

$$\tilde{\psi}^{(m)} = \exp(i\varphi)[E^{(m)} - \mathscr{E}]^{-1/2} A^- \psi^{(m)} \tag{44b}$$

$$\psi^{(m)} = \exp(-i\varphi)[E^{(m)} - \mathscr{E}]^{-1/2} A^+ \tilde{\psi}^{(m)} \tag{44c}$$

with

$$A^{\pm}(\mathscr{E}) = \frac{1}{\sqrt{2}} \left[-\frac{d}{dx} + \left(\frac{d}{dx} \ln \xi(x, \mathscr{E}) \right) \right]. \tag{44d}$$

The phase factor φ has been left undetermined. Furthermore, the non-normalisable solutions ψ and $\tilde{\psi}$ for energy $\mathscr{E}$ are connected by

$$\tilde{\psi}(x, \mathscr{E}) = 1/\psi(x, \mathscr{E}). \tag{45}$$

In this section, we assume that $-\infty \leqslant x \leqslant \infty$ and postpone discussion of $0 \leqslant r \leqslant \infty$ to II. It is necessary to make this distinction because the types of singularities of the potential V that are physically admissible depends upon the range of values of the variable x. Potentials with singularities $1/r^2$ are admissible for the radial problem, but a $1/x^2$ singularity is inadmissible when $-\infty \leqslant x \leqslant \infty$. Furthermore, the discussion of the construction of a normalisable $1/\psi$ depends upon the spatial domain in which ψ and V are defined. We now examine the question of the normalisability of $1/\psi$ when $-\infty \leqslant x \leqslant \infty$.

Let $\varphi_1(x, \mathscr{E})$ be a nodeless solution of $H\psi = E\psi$ for $\mathscr{E} < E^{(0)}$. Another linearly independent solution at the same energy is given by

$$\varphi_2(x, \mathscr{E}) = \varphi_1(x, \mathscr{E}) \int^x dz/\varphi_1^2(z, \mathscr{E}). \tag{46}$$

The nodelessness of φ_1 guarantees that this integral is well defined. The general solution at energy $\mathscr{E}$ is given by

$$\psi(x, \mathscr{E}, \alpha) = \varphi_1 + \alpha\varphi_2 = \varphi_1 \left(1 + \alpha \int_{-\infty}^x (dz/\varphi_1^2) \right) \tag{47}$$

in which the lower limit on the integral has been chosen to be $-\infty$ and α is an arbitrary constant. For potentials that vanish in the asymptotic region, when $\mathscr{E} < E^{(0)}$, φ_1 grows

exponentially, as $|x| \to \infty$ and φ_2 vanishes at $x = -\infty$ but grows exponentially as $x \to +\infty$. Let

$$\beta(\mathscr{E}) = \left(\int_{-\infty}^{\infty} dz/\varphi_1^2(x, \mathscr{E}) \right)^{-1}. \tag{48}$$

For values of α in the range $-\beta < \alpha < \infty$, ψ will remain nodeless, though $1/\psi$ is unnormalisable. Therefore, for $-\beta < \alpha < \infty$, $1/\psi$ is singularity free and normalisable because $1/\psi$ decreases exponentially as $|x| \to \infty$. This range of values of α then leads to normalisable $1/\psi$ and $1/\psi$ corresponds to an eigenstate of H as defined in equation (40) with ground state $E^{(0)} = \mathscr{E}$.

For the limiting values $\alpha = -\beta$ and $\alpha = \infty$, ψ would vanish exponentially at $x = \pm\infty$ and grow exponentially $x = \mp\infty$. Therefore for these limiting values $1/\psi$ would become unnormalisable. We first consider $\alpha = -\beta$ and denote the solution ψ for this value of α by ξ. Then

$$\xi \sim \varphi_1 \int_{+\infty}^{x} (dz/\varphi_1^2). \tag{49}$$

Since for $\mathscr{E} = -\frac{1}{2}\gamma^2$ the asymptotic behaviour of φ_1 is given by

$$\lim_{|x| \to \infty} \varphi_1 \sim \exp(\gamma|x|), \tag{50}$$

the asymptotic behaviour of ξ can be inferred to be

$$\lim_{x \to -\infty} \xi \sim \exp(-\gamma x) \qquad \text{and} \qquad \lim_{x \to +\infty} \xi \sim \exp(-\gamma x). \tag{51}$$

Therefore

$$\lim_{|x| \to \infty} \frac{d^2}{dx^2} \ln \xi(x, \mathscr{E}) \sim 0. \tag{52}$$

Similarly, for $\alpha = +\infty$,

$$\psi(x, \mathscr{E}, \alpha = \infty) \equiv \xi' \sim \varphi_1 \int_{-\infty}^{x} (dz/\varphi_1^2) \tag{53a}$$

$$\lim_{|x| \to \infty} \xi' \sim \exp(\gamma x). \tag{53b}$$

Hence

$$\lim_{|x| \to \infty} \frac{d^2}{dx^2} \ln \xi'(x, \mathscr{E}) \sim 0. \tag{54}$$

Though ψ vanishes at $x = \pm\infty$ for these limiting values of α, $(d^2/dx^2) \ln \psi(x, \mathscr{E}, \alpha)$ remains finite at $x = \pm\infty$.

Thus, for the limiting values $\alpha = -\beta$ and $\alpha = \infty$, though $1/\psi(x, \mathscr{E}, \alpha)$ is unnormalisable, $(d^2/dx^2) \ln \psi$ is well behaved, i.e. divergence free and finite in the asymptotic region. These values of α then lead to $\tilde{H}$ defined in equation (43) with the same spectrum as H. But, if $\alpha < -\beta$, ψ vanishes for some finite value of x, as can be seen from an examination of equation (47) and $(d^2/dx^2) \ln \psi(x, \mathscr{E}, \alpha)$ then diverges when ψ vanishes. Hence, for $\alpha < -\beta$, $\psi(x, \mathscr{E}, \alpha)$ does not lead to a physically acceptable potential $\tilde{V}$. The above analysis is illustrated with examples in the next section.

5.1. Free particle

Let $V(x)=0$. H has only a positive energy spectrum. For negative energies $\mathscr{E}=-\frac{1}{2}\gamma^2$, the general solution of $H\psi=\mathscr{E}\psi$ is given by

$$\psi(x,\mathscr{E})=\cosh\gamma x+\alpha\sinh\gamma x. \tag{55}$$

Though ψ is unnormalisable, for values of the parameter α in the range $|\alpha|<1$, ψ is nodeless and $1/\psi$ is normalisable. The family of potentials

$$\tilde{V}=V-\frac{d^2}{dx^2}\ln\psi(x,\mathscr{E})=\frac{\gamma^2(1-\alpha^2)}{(\cosh\gamma x+\alpha\sinh\gamma x)^2} \qquad |\alpha|<1 \tag{56}$$

therefore have a single bound state at energy

$$\tilde{E}^{(0)}=-\tfrac{1}{2}\gamma^2 \tag{57a}$$

with ground-state eigenfunctions

$$\tilde{\psi}^{(0)}\sim\frac{1}{\psi}=\frac{1}{(\cosh\gamma x+\alpha\sinh\gamma x)} \qquad |\alpha|<1. \tag{57b}$$

For positive energies, equation (44) then gives

$$\tilde{\psi}(x,E)=-[2(E-\tilde{E}^{(0)})]^{-1/2}\left(-\frac{d}{dx}+\gamma\frac{\sinh\gamma x+\alpha\cosh\gamma x}{\cosh\gamma x+\alpha\sinh\gamma x}\right)\psi(x,E). \tag{58}$$

In the asymptotic region $|x|\to\infty$, this equation becomes

$$\lim_{|x|\to\infty}\tilde{\psi}(x,E)=-[2(E-\tilde{E}^{(0)})]^{-1/2}\left(-\frac{d}{dx}+\gamma\right)\lim_{|x|\to\infty}\psi(x,E). \tag{59}$$

The α independence of this equation means that the transmission coefficient of this family of potentials $V(x,E,\alpha)$ are identical. This family of potentials is an example of the 'phase equivalent' family of Bargmann.

5.2. Simple harmonic oscillator

The oscillator potential does not belong to the category of potentials that remain finite in the asymptotic region. Nevertheless, the oscillator example serves to clarify some of the discussion in § 5.

The harmonic oscillator Hamiltonian

$$H=-\frac{1}{2}\frac{d^2}{dx^2}+\tfrac{1}{2}x^2 \tag{60}$$

has the eigenvalue spectrum

$$E=(n+\tfrac{1}{2}) \qquad n=0,1,2,\ldots. \tag{61}$$

The even solution of $H\psi=\mathscr{E}\psi$ for all energies can be written in series form (Abromowitz and Stegun 1965) and is given by

$$\varphi_1(x,\mathscr{E})=(\exp-\tfrac{1}{2}x^2)\left(1+\delta x^2+\delta(4+\delta)\frac{x^4}{4!}+\delta(4+\delta)(8+\delta)\frac{x^6}{6!}+\ldots\right) \tag{62a}$$

with

$$\delta=(1-2\mathscr{E}). \tag{62b}$$

For energies below the ground state of the oscillator $\mathscr{E} < E^{(0)} = \frac{1}{2}$, $\delta > 0$, which guarantees that φ_1 is positive definite. φ_1 is then a nodeless function of x. $\varphi_1(x, \mathscr{E})$ can also be written in terms of the standard parabolic cylinder functions U and V as

$$\varphi_1(x, \mathscr{E}) = 2^{-(\frac{1}{2}\mathscr{E}+\frac{3}{4})} \frac{\Gamma(\frac{3}{4}-\frac{1}{2}\mathscr{E})}{\pi^{1/2}} \left((1+\sin \pi\mathscr{E}) U(-\mathscr{E}, x\sqrt{2}) + \frac{\pi V(-\mathscr{E}, x\sqrt{2})}{\Gamma(\frac{1}{2}-\mathscr{E})} \right). \tag{62c}$$

The asymptotic expressions for U and V show that

$$\lim_{|x|\to\infty} \varphi_1(x, \mathscr{E}) \sim x^{-(\mathscr{E}+\frac{1}{2})} \exp(\tfrac{1}{2}x^2). \tag{63}$$

Thus $\varphi_1(x, \mathscr{E})$ is a nodeless unnormalisable solution when $\mathscr{E} < \frac{1}{2}$. The linearly independent solution

$$\varphi_2(x, \mathscr{E}) = \varphi_1 \int_0^x \mathrm{d}z / \varphi_1^2(z) \tag{64}$$

can also be written in series form as

$$\varphi_2(x, \mathscr{E}) = x(\exp -\tfrac{1}{2}x^2) \left(1 + \frac{x^2}{2!}(2+\delta) + \frac{x^4}{5!}(2+\delta)(6+\delta) + \ldots \right). \tag{65a}$$

φ_2 vanishes at $x = 0$, but the series within the parentheses is positive definite when $\mathscr{E} < \frac{1}{2}$. In terms of the parabolic cylinder functions

$$\varphi_2(x, \mathscr{E}) = 2^{-(\frac{1}{2}\mathscr{E}+\frac{7}{4})} \frac{\Gamma(\frac{1}{4}-\frac{1}{2}\mathscr{E})}{\pi^{1/2}} \left((\sin \pi\mathscr{E} - 1) U(-\mathscr{E}, x\sqrt{2}) + \frac{\pi V(-\mathscr{E}, x\sqrt{2})}{\Gamma(\frac{1}{2}-\mathscr{E})} \right). \tag{65b}$$

$$\lim_{|x|\to\infty} \varphi_2(x, \mathscr{E}) \sim x^{-(\mathscr{E}+\frac{1}{2})} \exp(\tfrac{1}{2}x^2). \tag{66}$$

The general solution at energy $\mathscr{E}$ is then given by

$$\psi(x, \mathscr{E}) = \varphi_1(x, \mathscr{E}) + \alpha\varphi_2(x, \mathscr{E}). \tag{67}$$

If we define the parameter

$$\beta(\mathscr{E}) = \left(\int_0^\infty \mathrm{d}z / \varphi_1^2(z, \mathscr{E}) \right)^{-1} = 2\frac{\Gamma(\frac{3}{4}-\frac{1}{2}\mathscr{E})}{\Gamma(\frac{1}{4}-\frac{1}{2}\mathscr{E})} \tag{68}$$

then, for $|\alpha| < \beta$,

$$\psi(x, \mathscr{E}, \alpha) = \varphi_1(x, \mathscr{E}) \left(1 + \alpha \int_0^x \mathrm{d}z / \varphi_1^2(z, \mathscr{E}) \right) \tag{69}$$

is nodeless and $1/\psi$ is normalisable. The family of Hamiltonians

$$\tilde{H} = H - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \psi(x, \mathscr{E}, \alpha) \qquad |\alpha| < \beta \tag{70}$$

therefore have identical spectra. This family of potentials is another example of the Bargmann 'phase equivalent' family.

In explicit terms:

$$\tilde{V}(x, \mathscr{E}, \alpha) = \tfrac{1}{2}x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln[\varphi_1(x, \mathscr{E}) + \alpha\varphi_2(x, \mathscr{E})] \qquad |\alpha| < \beta(\mathscr{E}), \qquad \mathscr{E} < \tfrac{1}{2} \tag{71}$$

have the spectra

$$\tilde{E}^{(0)} = \mathscr{E} < \tfrac{1}{2} \tag{72a}$$

$$\tilde{E}^{(m)} = (m - \tfrac{1}{2}) \qquad m = 1, 2, \ldots \tag{72b}$$

with eigenfunctions

$$\tilde{\psi}_2^{(0)}(x) \sim [\varphi_1(x, \mathscr{E}) + \alpha\varphi_2(x, \mathscr{E})]^{-1} \tag{72c}$$

$$\tilde{\psi}^{(m)}(x, \mathscr{E}, \alpha) = -(m - \tfrac{1}{2} - \mathscr{E})^{-1/2} A^-(\mathscr{E})\psi^{(m-1)}(x) \qquad m = 1, 2, \ldots \tag{72d}$$

where

$$A^-(\mathscr{E}) = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}x} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\ln(\varphi_1(x, \mathscr{E}) + \alpha\varphi_2(x, \mathscr{E}))\right)\right] \tag{72e}$$

and $\psi^{(J)}$ is a harmonic oscillator eigenfunction. Since the energy $\tilde{E}^{(0)}$ is arbitrary as long as $\tilde{E}^{(0)} < \frac{1}{2}$, the above equations give a recipe for constructing anharmonic potentials with spectra defined by equation (72).

The asymptotic behaviour of these potentials $V(x, \mathscr{E}, \alpha)$ can be studied analytically using the asymptotic form of the parabolic cylinder functions. It is easy to show that

$$\lim_{|x|\to\infty} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln(x, \mathscr{E}, \alpha) \sim 1. \tag{73}$$

Therefore

$$\lim_{|x|\to\infty} \tilde{V}(x, \mathscr{E}, \alpha) \sim \tfrac{1}{2}x^2 - 1. \tag{74}$$

This asymptotic limit is independent of α or $\mathscr{E}$. Furthermore

$$\lim_{x\to 0} \varphi_1(x, \mathscr{E}) = 1 \qquad \lim_{x\to 0} \frac{\mathrm{d}\varphi_1(x, \mathscr{E})}{\mathrm{d}x} = 0$$

$$\lim_{x\to 0} \varphi_2(x, \mathscr{E}) = 0 \qquad \lim_{x\to 0} \frac{\mathrm{d}\varphi_2(x, \mathscr{E})}{\mathrm{d}x} = 1. \tag{75}$$

Hence

$$\lim_{x\to 0} \tilde{V}(x, \mathscr{E}, \alpha) = 2\mathscr{E} + \alpha^2. \tag{76}$$

Using the series expansion for φ_1 and φ_2, the potentials $\tilde{V}(x, \mathscr{E}, \alpha)$ have been calculated for a range of values of E and $\alpha < \beta(\mathscr{E})$. Figure 3 shows $V(x, \mathscr{E}, \alpha)$ for $\mathscr{E} = -\frac{1}{2}$ and α in the range of values $0 < \alpha < 2/\sqrt{\pi}$. Since the potentials $\tilde{V}$ in equation (71) satisfy the condition $\tilde{V}(x, \mathscr{E}, \alpha) = V(-x, \mathscr{E}, -\alpha)$ only the results for positive values of α are shown. The potential for the corresponding negative values of α can be obtained by mirror reflection about the y axis. As remarked earlier, this family of potentials is an example of the 'phase equivalent' family. For $\alpha = 0$, $\tilde{V}(x) = \frac{1}{2}x^2 - 1$ is a shifted oscillator. This is the only value of α for which $\tilde{V}(x)$ is invariant under parity transformation. Thus, by imposing a specific condition on $\tilde{V}(x)$, a unique member of the family is obtained. Figures 4-7 show $\tilde{V}(x, \mathscr{E}, \alpha)$ for a range of value of $\mathscr{E}$ for a fixed value of $\alpha = 0$. These figures show that for $0 < \mathscr{E} < \frac{1}{2}$, the ground state of the new potential $\tilde{V}(x)$ lies inside a double well. This is an example of the general result that when $\mathscr{E}$ lies below the ground state of a given potential $V(x)$, but $\mathscr{E} > V_{\min}(x)$ where $V_{\min}(x)$ is the absolute minimum of the potential, the resulting partner potential $\tilde{V}(x)$ is necessarily a double well. It can be shown on general grounds that a double well is necessary to accommodate the new level at $\mathscr{E}$ close to the first excited state of $\tilde{V}(x)$ at energy $E^{(0)}$. Figures 4-7 show double well potentials whose exact spectrum is fixed by construction to be of the form indicated in equation (72).

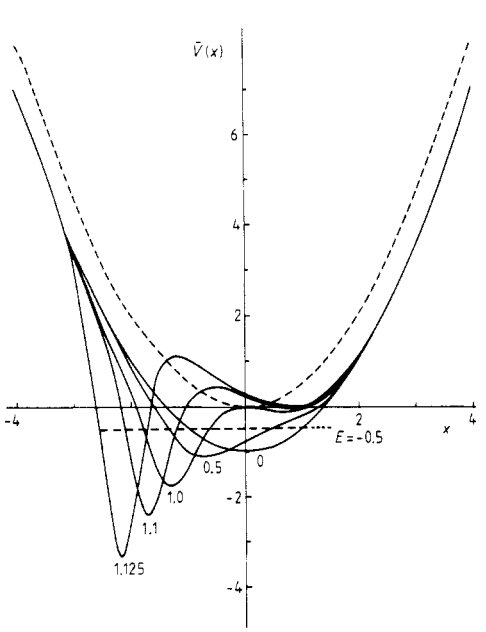

Figure 3. The 'phase equivalent' potentials $V(x, \mathscr{E}, \alpha)$ are shown for $\mathscr{E} = -\frac{1}{2}$. The value of α is indicated below each curve. The harmonic oscillator potential $V(x) = \frac{1}{2}x^2$ is also shown as a broken curve. The potentials shown in the figure by full curves have identical spectra with ground state at $\tilde{E}^{(0)} = -\frac{1}{2}$ as indicated by the horizontal broken line. The rest of the spectrum is identical to that of the oscillator.

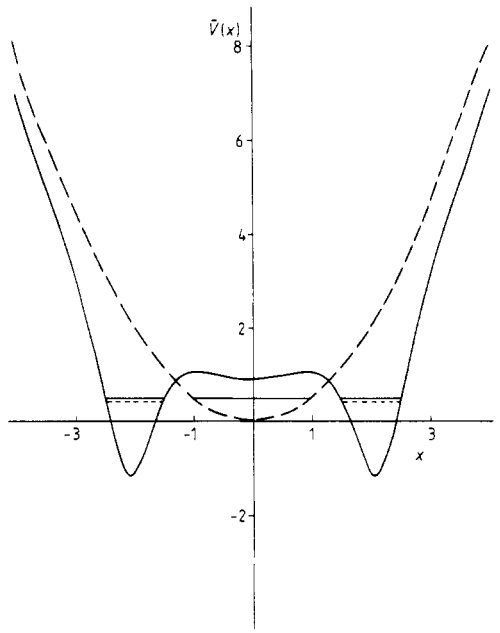

Figure 4. The potential $\tilde{V}(x, \mathscr{E}, \alpha)$ for $\mathscr{E} = 0.45$ and $\alpha = 0$. The harmonic oscillator potential is indicated as a broken curve. The ground state of $\tilde{V}$ at energy $\mathscr{E}$ is indicated as a broken line. The first excited state which is degenerate with the oscillator ground state is indicated by a full line. The rest of the spectrum of V is identical to that of the harmonic oscillator.

We next consider the limiting values $\alpha = \pm\beta$. Since

$$\psi(x, \mathscr{E}, \pm\beta) \sim U(-\mathscr{E}, \mp x\sqrt{2}) \tag{77a}$$

$$\lim_{x\to\infty} \psi(x, \mathscr{E}, \pm\beta) \sim \chi(\mp\mathscr{E} - \tfrac{1}{2}) \exp(\pm\tfrac{1}{2}x^2) \tag{77b}$$

$$\lim_{x\to-\infty} \psi(x, \mathscr{E}, \pm\beta) \sim \chi(\pm\mathscr{E} - \tfrac{1}{2}) \exp(\mp\tfrac{1}{2}x^2) \tag{77c}$$

the functions $1/\psi(x, \mathscr{E}, \pm\beta)$ are unnormalisable. But,

$$\lim_{x\to-\infty} \frac{d^2}{dx^2} \ln \psi(x, \mathscr{E}, \pm\beta) \sim \pm 1 \tag{78a}$$

$$\lim_{x\to+\infty} \frac{d^2}{dx^2} \ln \psi(x, \mathscr{E}, \pm\beta) \sim \mp 1 \tag{78b}$$

$$\lim_{x\to 0} \frac{d^2}{dx^2} \ln \psi(x, \mathscr{E}, \pm\beta) = -2\mathscr{E} - (\beta(\mathscr{E}))^2. \tag{78c}$$

These limiting values show that $\lim_{|x|\to\infty} (d^2/dx^2) \ln \psi(x, \mathscr{E}, \beta(\mathscr{E}))$ is divergence free even though $\psi(x, \mathscr{E}, \beta)$ vanishes at $x = \mp\infty$. Hence

$$\tilde{V}(\mathscr{E}) = \tfrac{1}{2}x^2 - \frac{d^2}{dx^2} \ln \psi(x, \mathscr{E}, \pm\beta(\mathscr{E})) \tag{79}$$

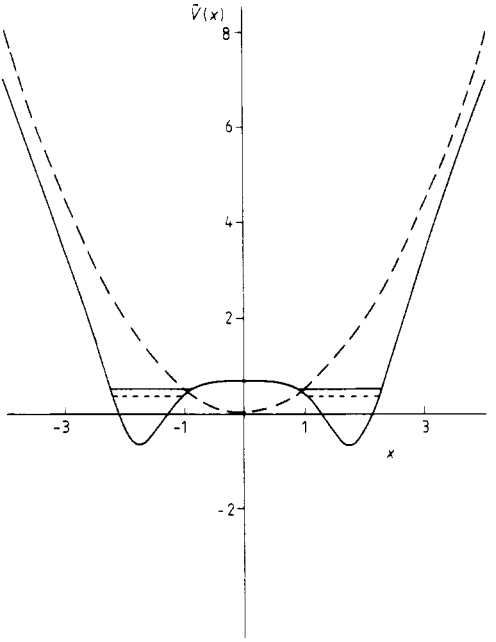

Figure 5. Same as figure 4, but for $\mathscr{E} = 0.35$.

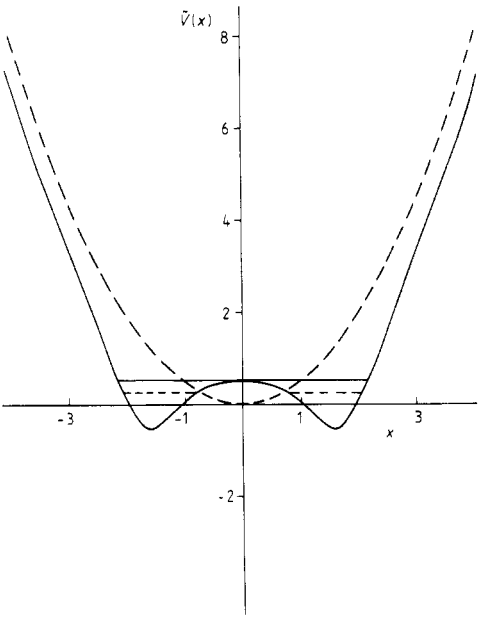

Figure 6. Same as figure 4, but for $\mathscr{E} = 0.25$.

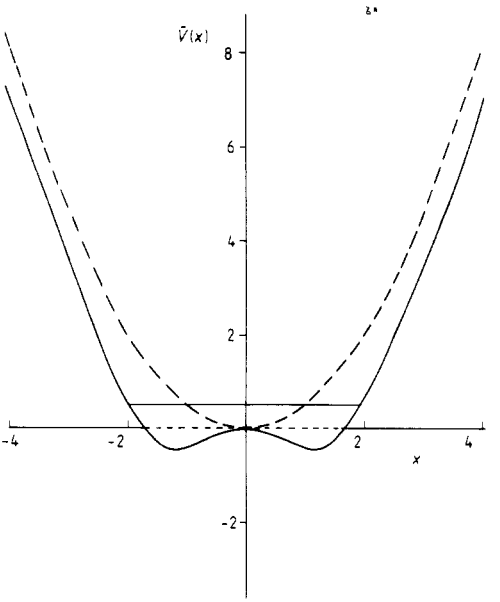

Figure 7. Same as figure 4, but for $\mathscr{E} = 0$.

has the spectrum

$$\tilde{E}^{(m)} = (m + \tfrac{1}{2}) \qquad m = 0, 1, 2, \ldots \tag{80a}$$

which is identical to the spectrum of the harmonic oscillator. The eigenfunctions of

$\tilde{H}$ are given in terms of the oscillator eigenfunctions by

$$\tilde{\psi}^{(m)}(x, \mathscr{E}) = (2m+1-2\mathscr{E})^{-1/2}\left[-\frac{\mathrm{d}}{\mathrm{d}x}+\left(\frac{\mathrm{d}}{\mathrm{d}x}\ln\psi(x, \mathscr{E}, \beta(\mathscr{E}))\right)\right]\psi^{(m)}(x). \tag{80b}$$

Therefore the Hamiltonian $\tilde{H}(\mathscr{E})$ for various values of $\mathscr{E} < \frac{1}{2}$ have spectra identical to the harmonic oscillator spectrum. They do not belong to the phase equivalent family, but $\tilde{V}(\mathscr{E})$ can be identified as members of the Bargmann class of potentials. This is discussed fully in II. The potentials $V(x, \mathscr{E}, \beta(\mathscr{E}))$ have been calculated numerically by the same procedure as that for $V(x, \mathscr{E}, \alpha)$. Figure 8 shows the potentials so calculated for a range of values of $\mathscr{E}$ and positive value for $\beta(\mathscr{E})$. The potential for negative β can be obtained by mirror reflection. The limiting values of $\tilde{V}$

$$\lim_{x\to\infty} \tilde{V}(x, \mathscr{E}, \beta(\mathscr{E})) \sim \tfrac{1}{2}x^2 - 1 \tag{81a}$$

$$\lim_{x\to-\infty} \tilde{V}(x, \mathscr{E}, \beta(\mathscr{E})) \sim \tfrac{1}{2}x^2 + 1 \tag{81b}$$

$$\lim_{x\to 0} \tilde{V}(x, \mathscr{E}, \beta(\mathscr{E})) = 2\mathscr{E} + \beta^2(\mathscr{E}) \tag{81c}$$

show that the leading order asymptotic limits of $V(x, \mathscr{E}, \beta(\mathscr{E}))$ are independent of $\mathscr{E}$.

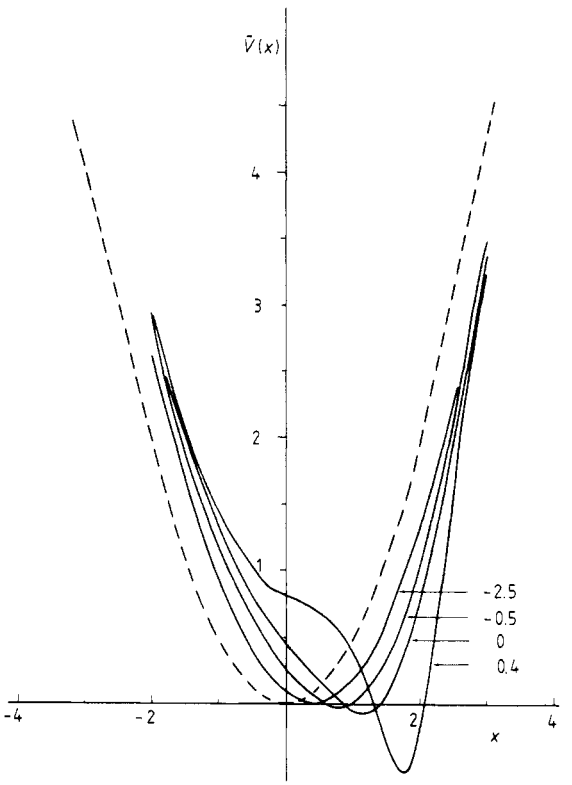

Figure 8. The potential $V(x, \mathscr{E}, \beta(\mathscr{E}))$ for a range of values of $\mathscr{E}$. The $\mathscr{E}$ value is indicated on each curve. The harmonic oscillator potential is shown as a broken curve. All the potentials shown in this diagram have spectra identical to that of the harmonic oscillator. The asymptotic values of the full curves are given by $\lim_{x\to\infty} \Delta V = -1$, $\lim_{x\to-\infty} \Delta V = +1$ where $\Delta V = \tilde{V}(x) - V(x)$.

When $|\alpha| > \beta$, $\psi(x, \mathscr{E}, \alpha)$ has a node at finite value of x which would lead to infinite barriers in the potential $\tilde{V}(x)$ for finite x. This is because $(d^2/dx^2) \ln \psi$ would diverge wherever ψ vanishes. Such potentials can be rejected on physical grounds.

6. Conclusions

In this paper it has been demonstrated that the algebra of supersymmetry can be used to find a partner Hamiltonian to any one-dimensional Hamiltonian of the Schrödinger equation. The flexibility in the choice of the partner Hamiltonian enables the identification of different types of 'supersymmetric' pairings. A procedure for constructing Hamiltonians either with identical spectra or with identical spectra, apart from a missing ground state, has been given. This recipe can be used to either add a new ground-state eigenvalue to, or eliminate the ground-state eigenvalue of, or maintain the same spectrum as a given Schrödinger Hamiltonian. This procedure may be repeated again and again in a suitable combination to generate hierarchies of Hamiltonians whose spectra are related to each other. By applying this procedure to the simple harmonic oscillator, anharmonic potentials whose spectra are identical to that of the harmonic oscillator or contain a ground state lower in eigenvalue than the ground state of the oscillator have been constructed. When the additional state has energy $0 < \tilde{E}^{(0)} < \frac{1}{2}$, the resulting anharmonic potential is shown to be a double well which accommodates the ground state inside the double well. Several recent papers have turned to the Gelfand–Levitan procedure of the inverse scattering method to generate anharmonic potentials (Abraham and Moses 1980, Mielnik 1984). In II, by applying the concept of a supersymmetric partner to a given radial Schrödinger equation, we establish the connection of the procedure outlined in this paper to the Gelfand–Levitan method (Gelfand and Levitan 1951) and show that the families of potentials generated by the 'supersymmetric method' belong to the Bargmann class of potentials (Bargmann 1949).

Appendix

In this appendix we study the nonlinear equation

$$U^2 + dU/dx = 2(V - \mathscr{E}). \tag{A1}$$

This equation can be solved by finding solutions of the auxiliary equation

$$d^2\psi/dx^2 = 2(V - \mathscr{E})\psi. \tag{A2}$$

In the main text, it was shown that $U = (d/dx) \ln \psi$ is a solution of equation (A1). However, a more general solution may easily be constructed (Nieto 1984). Let

$$f = \frac{d \ln \psi}{dx} \qquad U = f + \varphi \tag{A3}$$

then equation (A1) shows that φ must satisfy

$$d\varphi/dx + \varphi^2 + 2f\varphi = 0. \tag{A4}$$

Let $y = 1/\varphi$, then

$$dy/dx - 2fy = 1 \tag{A5}$$

leads to the solution

$$y=\left(\exp\int^{x}2f(z)\,dz\right)\left[\lambda+\int^{x}\left(\exp-\int^{z}2f(y)\,dy\right)dz\right] \tag{A6}$$

where λ is an arbitrary constant. The complete solution for U is then given by

$$U=\frac{d}{dx}\ln\psi+\frac{1/\psi^2}{(\lambda+\int^{x}dz/\psi^2(z))} \tag{A7}$$

which is the equation used in the main text. U may also be written as

$$U=\frac{d}{dx}\ln\left(\lambda\psi+\psi\int^{x}dz/\psi^2(z)\right). \tag{A8}$$

It is clear now that this solution could have been obtained by examination of the differential equation (A2). If ψ_1 is a solution of this equation, the linearly independent solution $(\psi_1\int^{x}dz/\psi_1^2(z))$ may be used to construct the general solution

$$\psi=\psi_1\left(\lambda+\int^{x}dz/\psi_1^2(z)\right). \tag{A9}$$

Since for any function $g(x)$

$$\frac{d^2\ln g}{dx^2}=\frac{1}{g}\frac{d^2g}{dx^2}-\left(\frac{1}{g}\frac{dg}{dx}\right)^2 \tag{A10}$$

the identification $U=(d/dx)\ln g$ then shows that equation (A1) may be used to write

$$\frac{1}{g}\frac{d^2g}{dx^2}=2(V-\mathscr{E})=\frac{1}{\psi}\frac{d^2\psi}{dx^2} \tag{A11}$$

with the immediate solutions $g=\psi$ and

$$U(x,\mathscr{E},\lambda)=\frac{d}{dx}\ln\left[\psi_1(x,\mathscr{E})\left(\lambda+\int^{x}dz/\psi_1^2(z,\mathscr{E})\right)\right] \tag{A12}$$

where we have used the general solution ψ in equation (A9).

It is now apparent that the equation for $(d/dx)\ln\psi$ is nonlinear as a consequence of the fact that the equation for ψ is a second-order linear differential equation and permits linear superposition of two independent solutions. To accommodate this superposition principle for ψ, the equation for $\ln\psi$ is nonlinear.

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